2-dimension Linear Least Squares

1. Suppose we believe that a variable z is dependent on two variables x, y via a linear relationship $z = ax + by + c$, and we are given n data points : $\left\{ \left(\begin{bmatrix} x_i \\ y_i \end{bmatrix}, z_i \right) : 1 \leq i \leq n \right\}$. How would you proceed to find a, b, c so as to minimize:

$$\sum_{i=1}^{n} (z_i - ax_i - by_i - c)^2?$$

2. Let $\mathbb{X} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ equipped with Binary addition structure. Consider the XOR (exclusive OR function) on $\mathbb{X}$, i.e

$$\text{XOR}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = 0, \quad \text{XOR}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 1, \quad \text{XOR}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = 1, \quad \text{XOR} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0.$$

The above is the true relationship but you are not told that. You are given the following data set of $\left(\begin{bmatrix} x \\ y \end{bmatrix}, z \right)$,

$$\left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right), \quad \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1 \right), \quad \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 \right), \quad \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0 \right) \right\}$$

(a) Assume z is a linear function of elements in $\mathbb{X}$. Find the best linear fit. (Note: *Take care to use Binary addition when applicable*)

(b) Let

$$W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, b = 0$$

and

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = w^T \left(\max\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, W^T \begin{bmatrix} x \\ y \end{bmatrix} + c \} \right) + b$$

i. Evaluate $h\left(\begin{bmatrix} x \\ y \end{bmatrix} \right)$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{X}$

ii. Evaluate $\sum_{i=1}^{4} \left(z_i - h\left(\begin{bmatrix} x_i \\ y_i \end{bmatrix} \right) \right)^2$

(c) In the previous question : can you device a procedure by which you can find W, w, c, b ?