1. For
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
in $\mathbb{R}^2$, we define
$$\langle x, y \rangle := y_1(2x_1 + x_2) + y_2(x_1 + x_2).$$
Show that this defines an inner product on $\mathbb{R}^2$.

2. *(*Cauchy-Schwarz Inequality*) Let $\langle , \rangle$ be an inner product on $\mathbb{R}^n$. Show that
$$|\langle x, y \rangle| \leq ||x||||y||,$$
for all $x, y \in \mathbb{R}^n$. Further, equality holds if and only if one is a multiple of the other (that is, x and y are linearly dependent).

3. (*Pythogoras Theorem*) Let $\langle , \rangle$ be an inner product on $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be a set of vectors such that they are pairwise orthogonal, that is, $\langle v_i, v_j \rangle = 0$ for all $i \neq j$, then
$$||\sum_{i=1}^{k} v_i||^2 = \sum_{i=1}^{k} ||v_i||^2.$$

4. Let $\langle , \rangle$ be an inner product on $\mathbb{R}^n$. Show that any set $S = (v_1, \ldots, v_k)$ of nonzero orthogonal vectors in $\mathbb{R}^n$ is linearly independent.

5. * Show that an $n \times n$ matrix A over $\mathbb{R}$ is orthogonal (ie. $AA^t = A^tA = I_n$) if and only if its columns form an orthonormal basis of $\mathbb{R}^n$ with respect to dot-product.